I'm having difficulty understanding what steps to take in applying valid argument forms to do a proof. What determines which given premises one should select to apply the valid argument forms – so that one can derive the intended conclusion? And how does one know which valid argument forms to apply?

Also, in Argument #1 (Lecture 11, Slide 11), the inference from steps 2 and 3 to 4 is stated as:

2. \( \neg y \)
3. \( j \rightarrow y \)
4. \( \neg j \) (from 2 and 3)

How can this be Modus Tollens? – given that Modus Tollens is actually as follow. Could anyone please assist?

\[
\begin{align*}
A \rightarrow B \\
\neg B \\
\cdots \\
\neg A
\end{align*}
\]

Hi Salvatore,

If you swap the order of 2 and 3, you will more easily see that the argument is a case of Modus Tollens.

3. \( j \rightarrow y \)
2. \( \neg y \)
4. \( \neg j \) (from 2 and 3)

The two formulas in steps 2 and 3 are premises used together to derive the formula in step 4. It does not actually matter in what order the two premises are laid out. It still has the same argument form, Modus Tollens.

**Example:** The following two arguments are one and the same:

I am older than you. You are older than her.
You are older than her. I am older than you.

\[
\begin{align*}
\text{I am older than her.} & \quad \text{I am older than her.} \\
\tau & \quad \tau
\end{align*}
\]

**Example:** The following four arguments are all Constructive Dilemma:

\[
\begin{align*}
A \lor B & \quad A \rightarrow C \\
A \lor B & \quad B \rightarrow D \\
A \lor B & \quad A \lor B \\
B \rightarrow D & \quad B \rightarrow D \\
A \lor B & \quad A \rightarrow C \\
A \lor B & \quad C \lor D \\
C \lor D & \quad C \lor D \\
C \lor D & \quad C \lor D
\end{align*}
\]

In the case of applying valid argument forms, the order in which the premises are laid out is not essential to the identity of an argument form. So long as all the premises used in an inference fit the forms of all the premises in a valid argument form and the conclusion of the inference also fits the form of the conclusion in the valid argument form, the inference fits that valid argument form.
About how to proceed in a proof using valid argument forms (a.k.a. natural deduction): Generally you select the premises that seem likely to generate some intermediate conclusions that seem likely generate the final conclusion that you need to prove.

Example: You are asked to prove by natural deduction that the following argument is valid.

\[ \begin{align*}
p & \& q \\
q & \rightarrow r \\
r & \rightarrow s \\
\hline
s
\end{align*} \]

The first thing to do is to lay out all the **given premises** in the following way:

1. \( p & q \) (given premise)
2. \( q \rightarrow r \) (given premise)
3. \( r \rightarrow s \) (given premise)

Now, think how you can get the intended conclusion, \( s \), by using from the information given in steps 1, 2, 3. A very useful strategy is to find a premise that already contains the conclusion you want to get, and then work backward.

Clearly, you want to get \( s \) (the intended conclusion). So, you look for a premise that contains \( s \). It is premise 3. Now, premise 3 can give you \( s \) by Modus Ponens - provided that you have \( r \) on its own. But there is no \( r \) on its own in the given premise. So, you need to generate \( r \) on its own first as an intermediate conclusion. How do you get \( r \) on its own? You look for a premise that contains \( r \) as a part. Ah! Premise 2 contains \( r \) as a part. But how can premise 2 give you \( r \) on its own? Premise 2 will give you \( r \) by Modus Ponens - provided that you have \( q \) on its own. But there is no \( q \) on its own. So how do you get \( q \) on its own? You look for another premise that contains \( q \). Ah! \( q \) is part of in premise 1 – which is a conjunction. Now, can premise 1 give you \( q \) on its own? Yes, it can, by Simplification, which says:

\[
\begin{align*}
A & \& B \\
\hline
A & B
\end{align*}
\]

So, you now write step 4 in the following way:

4. \( q \) (from 1, by Simp.)

And then, as we have seen before, from \( q \) (in step 4) together with \( q \rightarrow r \) (in step 2), you can get \( r \) by Modus Ponens. So you write step 5 in the following way:

5. \( r \) (from 2 and 4, by MP)

Again, as we have seen before, from \( r \) (in step 5) together with \( r \rightarrow s \) (in step 3), you can get \( s \) by Modus Ponens. So you write step 6 in the following way:

6. \( s \) (from 3 and 5, by MP)

So you get \( s \) eventually! And the proof ends here.

For more examples with explanation, please read Core Reading 1 for Week 11. Natural deduction is to some extent a trial and error process – especially for beginners. People get better at it after more practice and studying more examples given in the readings.

Best,
Norva
I'm just reviewing Lecture 11 and I understand the content as far as the use of natural deduction (use of valid argument forms to prove the validity of an argument). However when it comes to me knowing which rule to use when and going through each of the steps I keep getting stuck and cannot go through to the end because there seem to be so many possibilities.

Despite having these problems, when I look at all of your solutions it makes perfect sense and I can follow your method through completely.

So my question is by the end of this lecture, should we be able to complete the problems from start to finish without any prompts, or do we just need to understand the process involved and when shown a complete example explain how they got to that point (because I could do that).

Thanks :)

---

Dear Alexandra,

I am very glad that by the end of the lecture you understood the solutions to the exercises.

To respond to your questions:

(1) Generally, the purpose of using some examples in a lecture is only to introduce new concepts or new skills (e.g., the method of natural deduction).

(2) So, students are not expected to be able to work out the solutions to the examples in advance. They are only expected to follow the solution and understand it at the end. So, a student should feel good if they can understand the solution given in the lecture and the readings.

(3) About natural deduction in particular, you are right in pointing out that there are more than one way to solve a problem, and it is difficult to work out in advance which way to go. Sometimes I myself have trouble too seeing in advance which way to go. So, I would simply do it by trial and error.

(4) It is like playing a sport, e.g., football, you need to put the ball in the goal, but there can be more than one strategy, more than one way, to do that. How do you choose? You are standing in the middle of the field, and others are approaching you from different directions. What do you do with the ball? Where do you go next? Experienced players will tell you that in time after accumulating more experience of failure and success, you will develop a better sense of direction and judgement about how to respond. Mmm ... to new players, that can sound uninformative, for the experienced players after all did not give you a rule of thumb about what to do in every different situation. At best, the experienced player can show you examples of how other good players play, and show you some examples of what decision a good player would make in certain situations. And by watching more and more of these examples, and also by playing at the field yourself, you will gradually learn and develop a better sense and better skills.

(5) Now, doing natural deduction is the same thing. The examples in the lectures function to show what a good player would do. You are not asked to be like them in the beginning. However, it is hoped that after following and understanding those examples, you will learn a bit more, and then after practicing the questions in the quizzes and in the text book, your skill will be improved a bit more again.

(6) This is a first year subject, and so it will not demand students to be able to give solutions of the kind given in the examples in the lectures. In the exam, students may be asked to do filling in the blanks of missing steps in a natural deduction proof (like in some questions in some quizzes).
If students want to develop more sophisticated skills in the area, then they can do a more advanced level subject in logic and systematic thinking in their second year.

Best,
Norva

by ALEXANDRA  -  19 September 2009 9:55 PM

Thanks Norva, that is really helpful. It is definitely true that practice gives you that bit more confidence. I found after just going through the lecture and the two discussion board questions, by the time I got to the quiz questions I was a little bit more confident in how to approach them.

Thanks again :)


Consider the following argument - with 3 premises leading to the conclusion.

\[ p \rightarrow q \]
\[ q \rightarrow r \]
\[ \sim r \]

\[ \sim p \]

The argument is valid. Its validity can be proven via a series of applications of the valid argument forms introduced in Lectures 10 and 11. The proof can be laid out as follows:

1. \( p \rightarrow q \) [given premise]
2. \( q \rightarrow r \) [given premise]
3. \( \sim r \) [given premise]
4. \( p \rightarrow r \) [from 1 & 2, by Hypothetical Syllogism (HS)]
5. \( \sim p \) [from 3 & 4, by Modus Tollens (MT)]

In the above proof, we have applied HS and MT to show that the three given premises validly lead to the designated conclusion. The whole argument is valid because it contains two atomic sub-arguments (one from 1 & 2 to 3, and the other from 3 & 4 to 5), and each of them is valid.

The original argument can also be proven valid in the following alternative way:

1. \( p \rightarrow q \) [given premise]
2. \( q \rightarrow r \) [given premise]
3. \( \sim r \) [given premise]
4. \( \sim q \) [from 2 & 3, by Modus Tollens (MT)]
5. \( \sim p \) [from 1 & 4, by Modus Tollens (MT)]

In the above alternative proof, we have applied MT twice to show that the three given premises validly lead to the designated conclusion. The whole argument is valid because it contains two atomic sub-arguments (one from 2 & 3 to 4, and the other from 1 & 4 to 5), and each of them is valid.

Now, can you prove that the following argument is valid? - via a series of applications of the valid argument forms covered in Lectures 10 and 11.

\[ p \lor q \]
\[ p \rightarrow r \]
\[ \sim r \]

\[ q \]

Fill in the blanks!

Re: Application of valid argument forms (1)
by Julia - Sunday, 14 October 2012, 8:33 PM

1. \( p \lor q \) [given premise]
2. \( p \rightarrow r \) [given premise]
3. \( \sim r \) [given premise]
4. \( \sim p \) [from P2 & P3, by Modus Tollens]
5. \( q \) [from P1 & P4, by Disjunctive Syllogism]

Re: Application of valid argument forms (1)
by Norva Lo - Monday, 15 October 2012, 9:45 PM

Well done!
Consider the following argument - with 3 premises leading to the conclusion.

\[ p \rightarrow (q \land r) \]
\[ p \]
\[ \neg s \]

\[ \therefore r \lor \neg s \]

The argument is \textbf{valid}. Its validity can be proven via a series of applications of the valid argument forms introduced in Lectures 10 and 11. The \textbf{proof} can be laid out as follows:

1. \[ p \rightarrow (q \land r) \]  \hspace{1em} [given premise]
2. \[ p \]  \hspace{1em} [given premise]
3. \[ \neg s \]  \hspace{1em} [given premise]
4. \[ q \land r \]  \hspace{1em} [from 1 \& 2, by Modus Ponens (MP)]
5. \[ r \]  \hspace{1em} [from 4, by Simplification (Simp.)]
6. \[ r \land \neg s \]  \hspace{1em} [from 3 \& 5, by Conjunction (Conj.)]

In the above proof, we have applied MP, Simp. and Conj. to show that the three given premises validly lead to the designated conclusion. The whole argument is valid because it contains three atomic sub-arguments (one from 1 \& 2 to 3, another one from 4 to 5, yet another one from 3 \& 5 to 6), and each of them is valid.

Now, \textbf{can you prove that the following argument is valid?} - via a series of applications of the valid argument forms covered in Lectures 10 and 11.

\[ (p \lor q) \land r \]
\[ p \rightarrow h \]
\[ q \rightarrow b \]
\[ \neg h \]

\[ \therefore b \]

Remember to \textbf{number each step} in your proof, and indicate the \textbf{direction of inference} for each atomic sub-argument. Your proof should look like the following:

1. \[ (p \lor q) \land r \]  \hspace{1em} [given premise]
2. \[ p \rightarrow h \]  \hspace{1em} [given premise]
3. \[ q \rightarrow b \]  \hspace{1em} [given premise]
4. \[ \neg h \]  \hspace{1em} [given premise]
5. \[ ... \lor ... \]  \hspace{1em} [from ..., by ...]
6. \[ ... \lor ... \]  \hspace{1em} [from ..., \& ..., \& ..., by ...]
7. \[ b \]  \hspace{1em} [from \& ..., \& ..., by ...]

\textbf{Fill in the blanks!}
Consider the following argument:

"If Hume is right, then morality is founded on sympathy. If Kant is right, then morality is founded on reason. Therefore, Hume is right or Kant is right. Kant is indeed right! It follows that Hume is not right. So, we should conclude that morality is not founded on sympathy."

(a) Use different small letters to label all the atomic statements contained in the argument.

**Hints:** One atomic statement contained in the argument is "Hume is right". So we have: h = Hume is right. Another atomic statement contained in the argument is "morality is founded on sympathy". So, we have: s = Morality is founded on sympathy. Can you identify and label all the other atomic statements?

(b) Put all the premises, intermediate conclusions, and the final conclusion into logical formulas. List all the formulas in the order of the argument's direction of inferences - indicating which one is supposed to follow from which other ones.

**Hints:** The first premise in the argument is "If Hume is right, then morality is founded on sympathy", which should be symbolized as the logical formula: h → s. The last intermediate conclusion is "Hume is not right", which should be symbolized as: ~h. The final conclusion is "morality is not founded on sympathy", which should be symbolized as: ~s. Can you identify and label all the other premises and intermediate conclusion?

When the whole argument is put as an ordered list of formulas, it should look like the following:

1.  h → s
2.  ... → ...
3.  ... v ...
4.  h v ... (from 1 & ... & ...)
5.  ...
6.  ~h (from ... & ...)
7.  ~s (from 1 & 6)

Fill in the blanks above!

(c) How many atomic sub-arguments does the whole argument contain? For each of them, state its supposed direction of inference.

**Hint:** The last atomic sub-argument goes from 1 and 6 to 7. (Note: an atomic sub-argument contains only 1 inference).

(d) For each atomic sub-argument identified in part (c) above: (i) Check whether it fits any argument form covered in Lectures 10 and 11. (ii) If an atomic sub-argument fits one of those argument forms, name the argument form, and state whether it is valid or invalid. (iii) If an atomic sub-argument does not fit any of those argument forms, make a judgment of your own on whether it is valid or invalid.

**Hint:** The last atomic sub-argument fits one of the argument forms covered in Lectures 10 and 11. What is it? Is it valid or invalid? What about the other atomic sub-arguments? Do they fit any argument forms covered in Lectures 10 and 11? And are they valid or invalid?

(e) Is the whole argument valid? Give reasons for your answer.
c) Sub-arguments

There are 3 atomic sub arguments.

The first sub-argument is from 1 and 2 [and 3] to 4.

The second sub-argument is from 4 and 5 to 6.

The third sub-argument is from 1 and 6 to 7.

d) Argument forms

The first sub-argument, from 1 and 2 [and 3] to 4, is not in a form mentioned, but is valid. When combined with 3, this argument is almost in the constructive dilemma form.

**[Incorrect.** This sub-argument is actually invalid**]** - despite that fact that it looks very similar to CD. The difference is that in both the 2nd and 3rd premises, the antecedent and consequent are swapped when the sub-argument in question is compared to CD. That is what makes the sub-argument invalid. It is rather like swapping the antecedent and consequent in the valid form Modus Ponens (MP) which would give rise to the invalid form Affirming the Consequent (AC)]

More explanation: The sub-argument in question has the form:

\[A \lor B \\
C \rightarrow A \\
D \rightarrow B \\
\]

\[\therefore \\
C \lor D \\
\]

This argument form is invalid. For there are counterexamples to it. That is, not all arguments that fit the form are valid, so the argument form itself is invalid. Consider the following argument which fits the form:

Inspector Rex is male or female.
If Inspector Rex is a man, then Rex is male.
if Inspector Rex is a woman, then Rex is female.

\[
\begin{align*}
A &= \text{Inspector Rex is male.} \\
B &= \text{Inspector Rex is female.} \\
C &= \text{Inspector Rex is a man.} \\
D &= \text{Inspector Rex is a woman.} \\
\text{Now, the above argument fits the form in question, but it is invalid - because all its premises are true but its conclusion is false! This shows that the argument form itself is invalid.} \\
\end{align*}
\]

The second sub-argument is from 4 and 5 to 6, and is in the form of disjunctive syllogism, and is valid.

**[Incorrect.** The argument form here is actually not DS! And the actual argument form is invalid. You should try to give a counter-example of your own to the actual argument form - to show that it is invalid.]

The third sub-argument is from 1 and 6 to 7, and is in the form denying the antecedent. This argument form is invalid.

e) Validity

The entire argument is invalid, as at least one of the sub arguments has been proven to be invalid.
Consider the complex argument below:

"John is a pianist only if he has big hands. But John has big hands only if he has long fingers. So, if John is a pianist, then he has long fingers. However, John is not a pianist. For either John has a piano at home or he is not a pianist. But he has no piano at home. So, John does not have long fingers."

(a) There are 6 premises and/or intermediate conclusions and 1 final conclusion in the argument. But none of them are atomic statements. How many atomic statements appear in the argument? List all of them, and use different small letters to label them.

Hints: "John is a pianist" is one atomic statement. "John has big hands" is another atomic statement. p = John is a pianist, b = John has big hands. Identify and label the other atomic statements.

(b) Put the whole argument (including all the premises, intermediate conclusions, and the final conclusion) into logical formulas. Also indicate which formula follows from which other formulas.

Hints: The first premise "John is a pianist only if he has big hands" should be symbolized as: ~b → ~p. Symbolize the rest of the argument. The argument, when put into logical formulas, should look like the following:

1. ~b → ~p
2. ...
3. ... (intermediate conclusion, from 1 and 2)
4. ...
5. ...
6. ... (intermediate conclusion, from 4 and 5)
7. ... (final conclusion, from ... and ...)

Fill in the blanks!

Re: Symbolization and Natural deduction

how does this look?

1. ~b → ~p
2. ~f → ~b
3. p → f (intermediate conclusion, from 1 and 2)
4. ~p
5. ~o → ~p
6. ~o (intermediate conclusion, from 4 and 5)
7. ~f (final conclusion, from ?4 and ?6)

Re: Symbolization and Natural deduction

Your response is quite good indeed! See my comments below and you can post a revised answer to score more.

About 5: The most direct or straightforward symbolization is: o v ~p (although yours is logically equivalent to it).

About the sub-argument involving 4, 5, 6: You have put the formulas in the order of the corresponding statements given in the original argument. But note the position of the inference indicator "for". The sub-argument in question has the structure:

~p. For o v ~p. But ~o.

This indicates that o v ~p and ~o are used as reasons to support ~p (which is an intermediate conclusion). So, the structure of the sub-argument in question should be revised as:

4. o v ~p
5. ~o
6. ~p (intermediate conclusion, from 4 and 5)

About 7: The "from ... and ..." part is incorrect.

Can you work out the correct answers and rewrite the whole symbolized argument in logical form?

Hint: The whole argument is actually valid! And so is every sub-argument in it. So, step 7 is validly derived from two previous steps. Which two?
Re: Symbolization and Natural deduction
by Anna Jane Burns - Friday, 19 October 2012, 10:04 PM

a) Atomic statements:

p = john is a pianist
b = john has big hands
f = john has long fingers
o = john has a piano at home

There are four atomic statements in this argument.

b) Formulas

1. \( \neg b \rightarrow \neg p \)
2. \( \neg f \rightarrow \neg b \)
3. \( p \rightarrow f \) (equivalent to \( \neg f \rightarrow \neg p \), intermediate conclusion from 1 and 2 - valid inference)
4. \( o \lor \neg p \)
5. \( \neg o \)
6. \( \neg p \) (intermediate conclusion from 4 and 5, by Disjunctive Syllogism)

\[ \neg f \]
(final conclusion from 3 and 6, by Modus Tollens)

The whole argument is valid because all the inferences contained in it are valid.

---

Re: More from Question 4 in Quiz 9
by Norva Lo - Monday, 22 October 2012, 4:05 PM

Perfect!!
Consider the following argument:

"If Abelard is not praying, then he is seeing Heloise. If Abelard is seeing Heloise, then he is either in the church or in the garden. If he is in the church, then he is carrying a bible. But if he is in the garden, then he is carrying a bunch of flowers. Now, we know that Abelard is not praying. Therefore, he is seeing Heloise, which implies that he is either in the church or in the garden. It follows that he is carrying a bible or a bunch of flowers."

(a) Use different small letters to label all the atomic statements contained in the argument.

Hints: One atomic statement contained in the argument is "Abelard is praying". So, we have: \( p = \text{Abelard is praying} \). Another atomic statement contained in the argument is "Abelard is seeing Heloise". So we have \( h = \text{Abelard is seeing Heloise} \). Can you identify and label all the other atomic statements?

(b) Put all the premises, intermediate conclusions, and the final conclusion into logical formulas. List all the formulas in the order of the argument’s directions of inferences - indicating which one is supposed to follow from which other ones.

Hints: The first premise in the argument is "If Abelard is not praying, then he is seeing Heloise". This premise should be symbolized as the logical formula: \( \neg p \rightarrow h \). Can you identify and label all the other premises, intermediate conclusions, and also the final conclusion?

When the whole argument is put as an ordered list of formulas, it should look the following:

1. \( \neg p \rightarrow h \)
2. \( h \rightarrow (... \lor ...) \)
3. \( ... \rightarrow ... \)
4. \( ... \rightarrow ... \)
5. \( \neg p \)
6. \( h \) (from 1 & 5)
7. \( ... \lor ... \) (from 2 & ...)
8. \( ... \lor ... \) (from ... & ... & ...)

Fill in the blanks above!

(c) How many atomic sub-arguments does the whole argument contain? For each of them, state its supposed direction of inference.

Hints: The first atomic sub-argument goes from 1 and 5 to 6. Can you tell the directions of inference for the other atomic sub-arguments? (Note: an atomic sub-argument contains only 1 inference).

(d) For each atomic sub-argument identified in part (c) above: (i) Check whether it fits any argument form covered in Lectures 10 and 11. (ii) If an atomic sub-argument fits one of those argument forms, name the argument form, and state whether it is valid or invalid. (iii) If an atomic sub-argument does not fit any of those argument forms, make a judgment of your own on whether it is valid or invalid.

Hints: The first atomic sub-argument has the form Modus Ponens (MP). Is it valid or invalid? What about the other atomic sub-arguments? Do they fit any argument forms covered in Lectures 10 and 11? And are they valid or invalid?

(e) Is the whole argument valid? Give reasons for your answer.
b) The whole argument symbolized (premises, intermediate conclusions, and final conclusion in the original given order):

1. \( \sim p \rightarrow h \)
2. \( h \rightarrow (c \lor g) \)
3. \( c \rightarrow b \)
4. \( g \rightarrow f \)
5. \( \sim p \)
6. \( h \) (from 1 & 5)
7. \( c \lor g \) (from 2 & 6)
8. \( b \lor f \) (from 7 & 3 & 4)

c) All sub-arguments with their directions of inference indicated (order of premises, intermediate conclusions rearranged to show their directions of inference more easily):

P1 – (1) \( \sim p \rightarrow h \)
P2 – (5) \( \sim p \)
P3 – (6) \( h \) (from 1 & 5)
P4 – (2) \( h \rightarrow (c \lor g) \)
P5 – (7) \( c \lor g \) (from 2 & 6)
P6 – (3) \( c \rightarrow b \)
P7 – (4) \( g \rightarrow f \)
C – (8) \( b \lor f \) (from 7 & 3 & 4)

d) Arguments forms of the sub-arguments:

P1 – (1) \( \sim p \rightarrow h \)
P2 – (5) \( \sim p \)
P3 – (6) \( h \) (from 1 & 5)

**Modus Ponens – VALID**

P4 – (2) \( h \rightarrow (c \lor g) \)
P5 – (7) \( c \lor g \) (from 2 & 6)

This is actually in the form of **Modus Ponens - VALID** – because if we switch P1 & P2 and make \( h = A \) and \( (c \lor g) = B \) we get:

\[
A \rightarrow B \\
A \\
------- \\
B
\]

P6 – (3) \( c \rightarrow b \)
P7 – (4) \( g \rightarrow f \)
C – (8) \( b \lor f \) (from 7 & 3 & 4)

**Constructive Dilemma – VALID**

e) As each sub-argument is valid then the whole argument is valid as well